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ON THE PALINDROMIC DECOMPOSITION OF BINARY WORDS

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ABSTRACT

We prove a precise formula for the minimal number $K(n)$ such that every binary word of length n can be divided into $K(n)$ palindromes. Also we estimate the average number $\overline{K}(n)$ of palindromes composing a random binary word of the length n.

Keywords: binary word, palindrome, measure of symmetry, measure of asymmetry.

1. Introduction

The present note arose from the following problem proposed at International Mathematical Tournament of Towns [4], p.8: *Prove that every binary word of length 60 can be divided into 24 symmetric subwords and that the number 24 cannot be replaced by 14.* A word $w = a_0 \dots a_n$ is called *symmetric* if $a_i = a_{n-i}$ for all $i \le n$. For symmetric words we shall use a more poetic term *palindrome*. Let S be the set of nonempty binary words over the alphabet $\{a, b\}$ and S^1 be the set S with added the empty word. Observe that the set S^1 is a semigroup with respect to the operation of concatenation. The length of a word $w \in S^1$ will be denoted by $l(w)$. In particular, the empty word has length 0.

The above tournament task suggests three general problems:

(1) Given a word $w \in S$ find the minimal number $m(w)$ of palindromes whose product in S is equal to w (thus the number $m(w)$ can be thought as a measure of asymmetry of w);

(2) Given a positive integer n find the number $K(n) = \max\{m(w) : l(w) = n\}$ equal to the maximal asymmetry measure of the "worst" binary word of length n;

(3) Estimate the average asymmetry measure $\overline{K}(n) = 2^{-n} \sum\{m(w) : l(w) = n\}$ of a random binary word of length n.

It should be noted that the first two questions were considered in [1] and [2] while the last question was suggested to the author by O.Verbitsky. Observe that the above problems are consistent only for a two-letter alphabet: for every positive integer n the word $(abc)^n$ in the three-letter alphabet $\{a, b, c\}$ contains only trivial symmetric subwords.

For small numbers n it turned to be possible to calculate the numbers $K(n)$ by computer:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$K(n)$	1	2	2	2	2	3	3	4	4	4	5	5	5	6	6
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$K(n)$	6	6	7	7	8	8	8	8	9	9	10	10	10	10	11

This data allowed us to suggest and prove a precise formula for $K(n)$:

Theorem 1 *$K(n) = [\frac{n}{6}] + [\frac{n+4}{6}] + 1$ for every number $n \neq 11$ and $K(11) = 5$.*

The number $n = 11$ is exceptional and the word of length 11 destroying the uniformity is $w = aababbaabab$. The computer calculation shows that w is a unique word of length 11 (up to change $a \leftrightarrow b$ and reading the word from the right) with $m(w) = 5$.

Theorem 1 will be proved by induction whose base uses the computer calculation of $K(n)$'s for $n \leq 29$. Let us remark that the same values of $K(n)$ for $n \leq 29$ were independently obtained by Aleksandr Spivak [2] which also suggested a similar formula for $K(n)$.

Theorem 1 shows that the "worst" word of length n is very asymmetric: it cannot be divided into $< n/3$ palindromes. Next, we show that a random binary word also is far from being symmetric: it cannot be divided into $< n/11$ palindromes. Like in the case of asymmetry measure $K(n)$ of a "worst" word of length n, we start with computer calculation of the asymmetry measure $\overline{K}(n)$ of a random word of length n for small numbers n.

n	$\overline{K}(n)$	$\overline{K}(n)/n$	n	$\overline{K}(n)$	$\overline{K}(n)/n$	n	$\overline{K}(n)$	$\overline{K}(n)/n$
1	1.00	1.0000	8	2.33	0.2910	15	3.36	0.2239
2	1.50	0.7500	9	2.46	0.2734	16	3.50	0.2188
3	1.50	0.5000	10	2.61	0.2613	17	3.66	0.2152
4	1.75	0.4375	11	2.75	0.2502	18	3.81	0.2114
5	1.75	0.3500	12	2.91	0.2425	19	3.96	0.2084
6	2.06	0.3438	13	3.05	0.2349	20	4.11	0.2055
7	2.09	0.2991	14	3.20	0.2285	21	4.26	0.2030

This table will help us to estimate the limit $\overline{K} = \lim\limits_{n\to\infty} \frac{\overline{K}(n)}{n}$.

Theorem 2 *The limit $\overline{K} = \lim\limits_{n\to\infty} \frac{\overline{K}(n)}{n}$ exists, is equal to $\inf\limits_{n\in\mathbf{N}} \frac{\overline{K}(n)}{n}$ and can be estimated as $0.08781\ldots < \overline{K} \leq 0.2030\ldots$.*

To get the upper bound for $\overline{K}$ we use the results of computer calculation while the lower bound is proved by a subtle analytic argument. From the table we can expect that the exact value of $\overline{K}$ is close to $1/5$. It suggests that an average binary word w can be divided into $5/l(w)$ palindromes with average length 5.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into eight lemmas. We start from the upper bound. Let w_n denote the word $aabab(bbaaba)^n$ and put $m_0 = 2$, $m_1 = 3$, $m_2 = 3$, $m_3 = 3$, $m_4 = 4$, $m_5 = 4$.

Lemma 3 *For every $n \geq 0$ we have*
$m(w_n) \leq 2n + m_0$.
$m(w_n b) \leq 2n + m_1$.
$m(w_n bb) \leq 2n + m_2$.
$m(w_n bba) \leq 2n + m_3$.
$m(w_n bbaa) \leq 2n + m_4$.
$m(w_n bbaab) \leq 2n + m_5$.

Proof. For $n = 0$ the lemma results from the following decompositions:
$w_0 = (aa)(bab)$,
$w_0 b = (a)(aba)(bb)$,
$w_0 bb = (a)(aba)(bbb)$,
$w_0 bba = (aa)(b)(abbba)$,
$w_0 bbaa = (aa)(b)(abbba)(a)$,
$w_0 bbaab = (a)(aba)(bb)(baab)$.
Suppose that we have already proved the lemma for $n = k$. Then
$m(w_{k+1}) = m(w_k bba(aba)) \leq 2k + 3 + 1 = 2(k+1) + 2$
$m(w_{k+1} b) = m(w_k bbaa(bab)) \leq 2k + 4 + 1 = 2(k+1) + 3$
$m(w_{k+1} bb) = m(w_k bbaaba(bb)) \leq 2(k+1) + 2 + 1 = 2(k+1) + 3$
$m(w_{k+1} bba) = m(w_k bbaab(abba)) \leq 2k + 4 + 1 = 2(k+1) + 3$
$m(w_{k+1} bbaa) = m(w_k bbaababb(aa)) \leq 2(k+1) + 3 + 1 = 2(k+1) + 4$
$m(w_{k+1} bbaab) = m(w_k bbaabab(baab)) \leq 2(k+1) + 3 + 1 = 2(k+1) + 4$. □

The following two lemmas are proved by routine computer calculations.

Lemma 4
Let $u \in S, l(u) = 6, w \in \{(bbaaba)^2 bu, (bbaaba)bbaaabau, bbaaabababbaau\}$. Then one of the following conditions is satisfied:
1. $(\exists v', w' \in S) : w \in v'w'S^1, l(v') < 6$ *and* $m_{l(v')} + m(w') < (5 + l(v') + l(w'))/3$.
2. $(\exists v', w' \in S) : w \in v'w'S^1, l(v') < 6$, $m_{l(v')} + m(w') \leq (5 + l(v') + l(w'))/3$ *and* $l(v') + l(w') + 5$ *is a multiple of* 6.
3. $w \in \{(bbaaba)^3 b, (bbaaba)^2 bbaaaba, (bbaaba)bbaaabababbaa\}$.

Lemma 5 *Let $u \in S$, $12 \leq l(u) < 18$ and $w \in \{(bbaaba)^2 bu, (bbaaba)bbaaabau, bbaaabababbaau\}$. Then one of the following conditions is satisfied:*

1. *There exist words* $v', w' \in S$ *such that* $w = v'w', l(v') < 5$ *and* $m_{l(v')} + m(w') \leq [17/2 + l(u)/4]$.

2. $w \in \{(bbaaba)^2 bbaaababbbaaababba, (bbaaba)^3 bbaaababababba\}$.

Lemma 6 *Let* $w \in aS$, $l(w) = 6n$, $n \geq 3$. *Then one of the following conditions is satisfied:*

1. $(\exists w' \in S) : w \in w'S^1$ *and* $m(w') < l(w')/3$.

2. $(\exists w' \in S) : w \in w'S^1$ *and* $m(w') \leq l(w')/3$ *and* $l(w')$ *is a multiple of* 6.

3. $w \in \{w_{n-1}b, w_{n-2}bbaaaba, w_{n-3}bbaaababababbaa\}$.

Proof. For $n = 3$ the lemma can be proved by a computer calculation. Suppose that we have already proved the lemma for $n = k$. Consider a word w such that $l(w) = 6(k+1)$. If the conditions 1 or 2 does not hold for the word w then they fail for the word consisting of the first $6k$ letters of the word w. Hence there exists a word $u \in S$ such that $l(u) = 6$ and $w \in \{w_{k-3}(bbaaba)^2 bu, w_{k-3}(bbaaba)bbaaabau, w_{k-3}bbaaababababbaau\}$. Then Lemmas 2 and 3 imply that the condition 3 is satisfied. □

Lemma 7 *Let* $v \in S$, $l(v) = 6n + r$, $0 \leq n$, $0 \leq r < 6$ *and* $l(v) \neq 11$. *Then* $m(v) \leq 2n + [3/2 + r/4]$. *If* $l(v) = 11$ *then* $m(v) \leq 5$.

Proof. Remark that for $k = 6n + r$ we get $(k + 1)/3 \leq 2n + [3/2 + r/4] \leq (k + 4)/3$ and if $k = 11$ then $5 \leq (k + 4)/3$. Also remark that $x \leq 2n + [3/2 + r/4]$ for each positive integer $x < (k + 4)/3$. For $l(v) \leq 29$ the lemma is proved by the computer calculation, see the above table. Suppose that we have already proved the lemma for all words v with $l(v) \leq k$, where $k \geq 29$.

Let $l(v) = k + 1 = 6n + r$. Then $n \geq 5$. Without loss of generality we may suppose that $v \in aS$. We consider the following cases:

1. There exist words $v_1 \in S, v_2 \in S^1$ such that $v = v_1v_2$ and $m(v_1) < l(v_1)/3$. Then $m(v) \leq m(v_1) + m(v_2) < l(v_1)/3 + (l(v_2) + 4)/3 = (l(v) + 4)/3$. Therefore $m(v) \leq 2n + [3/2 + r/4]$.

2. There exist words $v_1, v_2 \in S$ such that $v = v_1v_2$, $m(v_1) \leq l(v_1)/3, l(v_2) \geq 12$ and $l(v_1) = 6t$. Then $m(v) \leq m(v_1)+m(v_2) \leq 2t+2(n-t)+[3/2+r/4] = 2n+[3/2+r/4]$.

3. The cases 1 and 2 do not hold. Let $v = v_1v_2$ where $l(v_1) = 6(n - 2), l(v_2) = 12 + r$. Then Lemma 5 implies that $v_1 \in \{w_{n-3}b, w_{n-4}bbaaaba, w_{n-5}bbaaababababbaa\}$. If there exist words $v', w' \in S$ such that $v_1v_2 = w_{n-5}v'w', l(v') < 5$ and $m_{l(v')} + m(w') \leq [17/2 + l(v_2)/4]$ then Lemma 2 implies that $m(v) \leq m(w_{n-5}v') + m(w') \leq 2(n - 5) + m_{l(v')} + m(w') \leq 2(n - 5) + [17/2 + 3 + r/4] = 2n + [3/2 + r/4]$. In the opposite case Lemma 4 applied to the last 25+r letters of the word w implies that $v_1v_2 \in \{w_{n-3}bbaaababbbaaab(abba), w_{n-2}bbaaabab(abba)\}$. Then $m(v) \leq ((l(v) - 4) + 4)/3 + 1 = l(v)/3 + 1 < (l(v) + 4)/3$ and hence $m(v) \leq 2n + [3/2 + r/4]$. □

The following lemmas will be uses to prove the lower bound.

Lemma 8 *For every* $n \geq 0$ *the word* $(bbaaba)^n$ *does not contain a palindrome* p *with* $l(p) \geq 5$.

Proof. Put $v = bbaaba$. If v^n contains a palindrome p with $l(p) \geq 2$, then v^n also contains a palindrome p' such that $l(p') = l(p) - 2$. Therefore it suffices to show that v^n does not contain a palindrome p with $l(p) \in \{5, 6\}$. Suppose the converse. Since the length of p does not exceed the length of v then we can find two consecutive subwords $v_1 = v_2 = v$ of v^n such that p is a subword of $v_1 v_2$. Thus v^2 also contains a palindrome p such that $l(p) \in \{5, 6\}$. The straight check shows the opposite. □

Lemma 9 *Let $n = 6t + 5 + r$, $t \geq 1$, $0 \leq r < 6$. Suppose that the word u_n consists of the first n letters of the word w_{t+1}. Then $m(u_n) = 2t + m_r$.*

Proof. Let $t \geq 1$ and $u_n = u_{n-k} p_k$, where p_k is a palindrome with $l(p_k) = k$. Lemma 6 implies that $k \leq 4$. Therefore the following cases are possible:

If $n = 6t + 5$ then $u_n = w_{t-1} bbaaba$. Hence $p_k = a$ or $p_k = aba$ and $m(u_{6t+5}) = \min\{m(u_{6t+4}), m(u_{6t+2})\} + 1$.

If $n = 6t + 6$ then $u_n = w_{t-1} bbaabab$. Hence $p_k = b$ or $p_k = bab$ and $m(u_{6t+6}) = \min\{m(u_{6t+5}), m(u_{6t+3})\} + 1$.

If $n = 6t + 7$ then $u_n = w_{t-1} bbaababb$. Hence $p_k = b$ or $p_k = bb$ and $m(u_{6t+7}) = \min\{m(u_{6t+6}), m(u_{6t+5})\} + 1$.

If $n = 6t + 8$ then $u_n = w_{t-1} bbaababba$. Hence $p_k = a$ or $p_k = abba$ and $m(u_{6t+8}) = \min\{m(u_{6t+7}), m(u_{6t+4})\} + 1$.

If $n = 6t + 9$ then $u_n = w_{t-1} bbaababbaa$. Hence $p_k = a$ or $p_k = aa$ and $m(u_{6t+9}) = \min\{m(u_{6t+8}), m(u_{6t+7})\} + 1$.

If $n = 6t + 10$ then $u_n = w_{t-1} bbaababbaab$. Hence $p_k = b$ or $p_k = baab$ and $m(u_{6t+10}) = \min\{m(u_{6t+9}), m(u_{6t+6})\} + 1$.

The computer calculation shows that $m(u_8) = 3$, $m(u_9) = 4$, $m(u_{10}) = 4$. Therefore $m(u_{11}) = 4$, $m(u_{12}) = 5$, $m(u_{13}) = 5$, $m(u_{14}) = 5$, $m(u_{15}) = 6$, $m(u_{16}) = 6$. This proves the lemma for $t = 1$.

Suppose that the lemma is already proved for $t = k$. Then for $t = k + 1$ we obtain:

$m(u_{6k+11}) = \min\{m(u_{6k+10}), m(u_{6k+8})\} + 1 = 2k + 4 = 2(k + 1) + m_0$.
$m(u_{6k+12}) = \min\{m(u_{6k+11}), m(u_{6k+9})\} + 1 = 2k + 5 = 2(k + 1) + m_1$.
$m(u_{6k+13}) = \min\{m(u_{6k+11}), m(u_{6k+12})\} + 1 = 2k + 5 = 2(k + 1) + m_2$.
$m(u_{6k+14}) = \min\{m(u_{6k+13}), m(u_{6k+10})\} + 1 = 2k + 5 = 2(k + 1) + m_3$.
$m(u_{6k+15}) = \min\{m(u_{6k+14}), m(u_{6k+13})\} + 1 = 2k + 6 = 2(k + 1) + m_4$.
$m(u_{6k+16}) = \min\{m(u_{6k+15}), m(u_{6k+12})\} + 1 = 2k + 6 = 2(k + 1) + m_5$. □

Lemma 10 *For every number $n \geq 0$ we have $m(aabab(bbaaba)^n bbaaababb) = 2n + 6$.*

Proof. For $n \leq 1$ the lemma is proved by the computer calculation. Let $n > 1$. Put $v_n = aabab(bbaaba)^n bbaa(ababb)$. We claim that if p is a palindrome such that $v_n = v' p v''$ and $l(v'') < 5$ then $l(p) \leq 5$. This can be proved by the straight check taking into account that for a palindrome p whose "center of symmetry" lies in the subword $(bbaaba)^n bb$ of the word v_n we can apply Lemma 7 to conclude that $l(p) \leq 4$.

Let $p_1 \dots p_{m(v_n)}$ be a decomposition of the word v_n, where $p_1, \dots, p_{m(v_n)}$ are palindromes. Take a number k such that $l(p_1 \dots p_k) \leq 6n + 5$ and $l(p_1 \dots p_{k+1}) > 6n + 5$. Put $v' = p_1 \dots p_{k+1}$, $v'' = p_{k+2} \dots p_{m(v_n)}$. Since $l(p_{k+1}) \leq 5$ then one of the following cases holds:

1. $v'' = baaababb$. Then Lemma 8 implies that $m(v') = 2n + m_1$ and the computer calculation shows that $m(v'') = 3$. Therefore $m(v') + m(v'') = 2n + m_1 + 3 = 2n + 6$.
2. $v'' = aaababb$. Then $m(v') = 2n + m_2$, $m(v'') = 3$. Therefore $m(v') + m(v'') = 2n + m_2 + 3 = 2n + 6$.
3. $v'' = aababb$. Then $m(v') = 2n + m_3$, $m(v'') = 3$. Therefore $m(v') + m(v'') = 2n + m_3 + 3 = 2n + 6$.
4. $v'' = ababb$. Then $m(v') = 2n + m_4$, $m(v'') = 2$. Therefore $m(v') + m(v'') = 2n + m_4 + 2 = 2n + 6$.

Hence $m(v_n) = m(v') + m(v'') = 2n + 6$. □

To finish the proof of Theorem 1 let us make the following remarks. Let $t = 6n + r$, $n \geq 0$, $0 \leq r < 6$ and $t \neq 11$. It is easy to verify that $[\frac{t}{6}] + [\frac{t+4}{6}] + 1 = 2n + [\frac{3}{2} + \frac{r}{4}]$. Lemma 6 implies that $K(t) \leq 2n + [3/2 + r/4]$. Lemma 8 implies that if $n \geq 2$ and $r \neq 2$ then $K(t) \geq 2n + [3/2 + r/4]$. Lemma 9 yields $K(t) \geq 2n + [3/2 + r/4]$ provided $n \geq 2$ and $r = 2$. Finally, the computer calculation shows that $K(11) = 5$ and $K(t) = 2n + [3/2 + r/4]$ provided $n \leq 1$ and $t \neq 11$.

3. Proof of the Theorem 2

We shall use the following *Subadditive Lemma* [3], §2.5.

Lemma *Let $\{x_n\}$ be a sequence such that $0 \leq x_{m+n} \leq x_m + x_n$ for every positive integer m, n. Then $\lim\limits_{n\to\infty} x_n/n = \inf\limits_{n\in\mathbf{N}} x_n/n$.*

To apply this lemma, observe that for positive integer n, m we have

$$\overline{K}(m+n) = 2^{-m-n}\sum\{m(w) : l(w) = m+n\} =$$

$$2^{-m-n}\sum\{m(w_1w_2) : l(w_1) = m, l(w_2) = n\} \leq$$

$$2^{-m-n}\sum\{m(w_1) + m(w_2) : l(w_1) = m, l(w_2) = n\} =$$

$$2^{-m}\sum\{m(w_1) : l(w_1) = m\} + 2^{-n}\sum\{m(w_2) : l(w_2) = n\} = \overline{K}(m) + \overline{K}(n).$$

Then the subadditive lemma yields the existence of the limit $\overline{K} = \lim\limits_{n\to\infty} \overline{K}(n)/n$ and an upper bound $\overline{K} \leq \frac{\overline{K}(21)}{21} = \frac{372487}{7\cdot 2^{18}} = 0.2030\ldots$.

Let $n \geq 9$. Next, we prove the lower bound for $\overline{K}$. Observe that $2^n\overline{K}(n) = \sum\limits_{k=1}^{K(n)} kx_k$, where $x_k = |\{w : l(w) = n, m(w) = k\}|$. In order to estimate the sum $\sum\limits_{k=1}^{K(n)} kx_k$, we shall use the following

Lemma 11 *Let $l \geq p$ and $x_1, \ldots, x_{l+1}, a_1, \ldots, a_{p+1}$ be nonnegative real numbers, $\sum\limits_{k=1}^{l+1} x_k = \sum\limits_{k=1}^{p+1} a_k$ and $x_k \leq a_k$ for all $1 \leq k \leq p$. Then $\sum\limits_{k=1}^{l+1} kx_k \geq \sum\limits_{k=1}^{p+1} ka_k$.*

Proof. Indeed, $\sum_{k=1}^{l+1} kx_k - \sum_{k=1}^{p+1} ka_k = \sum_{k=1}^{l+1} kx_k - \sum_{k=1}^{p} ka_k - (p+1)\left(\sum_{k=1}^{l+1} x_k - \sum_{k=1}^{p} a_k\right) = \sum_{k=1}^{l+1}(k-p-1)x_k + \sum_{k=1}^{p}(p+1-k)a_k = \sum_{k=p+1}^{l+1}(k-p-1)x_k + \sum_{k=1}^{p}(p+1-k)(a_k - x_k) \geq 0.$ □

Now we are going to find numbers a_k satisfying the conditions of Lemma 11. Let w be a word such that $m(w) = k$. Then $w = p_1 \cdots p_k$ for some palindromes $p_1, \ldots, p_k$. For a fixed decomposition $n = n_1 + \ldots + n_k$ as a sum of positive integers there exist $\prod_{i=1}^{k} 2^{\left[\frac{n_k+1}{2}\right]} \leq 2^{\frac{n+k}{2}}$ different products of palindromes $p_1, \ldots, p_k$ such that $l(p_i) = n_i$ for every i. Since there exist $\binom{n-1}{k-1}$ decompositions of n as a sum of k positive integer components then there exist not more than $a_k = \binom{n-1}{k-1} 2^{\frac{n+k}{2}}$ different products of k palindromes. Hence $x_k \leq a_k$.

In fact the estimation $x_k \leq a_k$ is too rough and there is a more subtle estimation: if $w = p_1 \ldots p_k$ for some palindromes $p_1, \ldots, p_k$ and $k < n/2$ then there exists a palindrome p_i such that $l(p_i) > 2$. Let $p_i = xp_i'x, x \in \{a, b\}$. Then there exists a decomposition $w = p_1 \ldots p_{i-1} x p_i' x p_{i+1} \ldots p_k$ of the word w as a product of $k+2$ palindromes. If $k+2 < n/2$ then there exists a decomposition of the word w as a product of $k+4$ palindromes and so forth. Since $K(n) < \frac{n}{2}$ for $n \geq 9$ we get $x_k \leq x_k + x_{k-2} + x_{k-4} + \ldots \leq a_k$ for $n \geq 9$ and $k \leq K(n)$.

There exists $p = p(n)$ such that $\sum_{k=1}^{p} a_k \leq \sum_{k=1}^{K(n)} x_k = 2^n$, $\sum_{k=1}^{p+1} a_k > 2^n$. For $1 \leq k \leq p$ put $\delta_k = \frac{a_k}{a_{k+1}} = \frac{(n-k-1)!k!}{\sqrt{2}(k-1)!(n-k)!} = \frac{k}{\sqrt{2}(n-k)}$. Since the sequence δ_k strictly increases then for all $l \leq k$ we have $a_l = a_{k+1}\delta_k\delta_{k-1}\cdots\delta_l \leq a_{k+1}\delta_k^{k+1-l}$. Since $p \leq K(n) < \frac{n}{2}$ for $n \geq 9$ then $\delta_k < \frac{1}{\sqrt{2}} < 1$ for every k. Therefore $2^n < \sum_{k=1}^{p+1} a_k \leq a_{p+1}(1 + \delta_p + \ldots + \delta_p^p) < \frac{a_{p+1}}{1-\delta_p}$. Hence $a_{p+1} = \binom{n-1}{p} 2^{\frac{n+p+1}{2}} > 2^n(1-\delta_p)$. Since $e^{\frac{1}{12m}}\sqrt{2\pi m}\left(\frac{m}{e}\right)^m > m! > \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$ (see 21.4-2 in [5]) for all positive integer m we obtain the estimation

$$e^{\frac{1}{12(n-1)}}\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1} 2^{\frac{n+p+1}{2}} > a_{p+1} > 2^n(1-\delta_p) >$$

$$\sqrt{2\pi p}\left(\frac{p}{e}\right)^p \sqrt{2\pi(n-1-p)}\left(\frac{n-1-p}{e}\right)^{n-1-p} 2^n(1-\delta_p),$$

that implies

$$\frac{1}{12(n-1)} + \frac{1}{2}\ln 2\pi(n-1) + (n-1)(\ln(n-1) - 1) + \frac{p+1-n}{2}\ln 2 >$$

$$\frac{1}{2}\ln 2\pi p + p(\ln p - 1) + \frac{1}{2}\ln 2\pi(n-1-p) + (n-1-p)(\ln(n-1-p) - 1) +$$

$$\ln(1-\delta_p).$$

Let $\theta_n = \frac{p(n)}{n-1}$. Put $r(n) = \frac{1}{12(n-1)} + \frac{1}{2}\ln 2\pi(n-1) - \frac{1}{2}\ln 2\pi p - \frac{1}{2}\ln 2\pi(n-1-p) - \ln(1-\delta_p)$. Then $r(n) = o(n-1)$ and

$$(n-1)(\ln(n-1)-1) + \frac{(\theta_n-1)(n-1)}{2}\ln 2 + r(n) >$$

$$\theta_n(n-1)(\ln\theta_n + \ln(n-1)-1)+$$

$$(n-1)(1-\theta_n)(\ln(1-\theta_n) + \ln(n-1)-1).$$

This implies that $f(\theta_n) > r(n)/(n-1)$, where $f(\theta) = \frac{\theta-1}{2}\ln 2 - \theta\ln\theta - (1-\theta)\ln(1-\theta)$, $f(0) = \lim\limits_{\theta\to+0} f(\theta) = -\frac{\ln 2}{2}$.

Let $\overline{\theta} = \overline{\lim\limits_{n\to\infty}}\theta_n$. By the continuity of the map f on $[0;1)$, we get $f(\overline{\theta}) = \overline{\lim\limits_{n\to\infty}} f(\theta_n) \geq \lim\limits_{n\to\infty} r(n)/(n-1) = 0$. Since $0 \leq \overline{\theta} \leq \lim\limits_{n\to\infty}\frac{K(n)}{n} = \frac{1}{3}$ and $f'(\theta) = \frac{\ln 2}{2} - \ln\theta + \ln(1-\theta) > 0$ for $0 < \theta \leq \frac{1}{3}$ we conclude that $\overline{\theta} \geq \theta'$, where θ' is the unique root of the equation $f(\theta) = 0$ on the segment $[0;\frac{1}{3}]$. Computer calculation shows that $\theta' = 0.09488\ldots$.

Using the inequalities $\sum\limits_{k=1}^{K(n)} x_k \leq 2^n$, $x_k \leq a_k$ for $k \leq p$, $a_{p-1} \leq \frac{2^n}{1+1/\delta_{p-1}}$, Lemma 11 and the equality $\delta_{p-1} = \frac{p-1}{\sqrt{2}(n-p+1)} = \frac{\theta_n}{\sqrt{2}(1-\theta_n)} + o(1)$ we obtain

$$\sum_{k=1}^{K(n)} kx_k \geq \sum_{k=1}^{p} ka_k + \left(2^n - \sum_{k=1}^{p} a_k\right)(p+1) = 2^n + \sum_{k=1}^{p}(k-1)a_k +$$

$$\left(2^n - \sum_{k=1}^{p} a_k\right)p \geq \left(2^n - \sum_{k=1}^{p-1} a_k\right)p \geq \left(2^n - a_{p-1}\sum_{k=1}^{p-1}\delta_{p-2}^{k-1}\right)p \geq$$

$$\left(2^n - \frac{a_{p-1}}{1-\delta_{p-2}}\right)p \geq \left(2^n - \frac{\frac{2^n}{1+1/\delta_{p-1}}}{1-\delta_{p-2}}\right)p \geq \left(2^n - \frac{\frac{2^n}{1+1/\delta_{p-1}}}{1-\delta_{p-1}}\right)p =$$

$$\left(1 - \frac{\delta_{p-1}}{1-\delta_{p-1}^2}\right)2^n p = \left(1 - \frac{\frac{\theta_n}{\sqrt{2}(1-\theta_n)}}{1 - \frac{\theta_n^2}{2(1-\theta_n)^2}} + o(1)\right)2^n p = g(\theta_n)2^n n + o(2^n n),$$

where $g(\theta) = \theta - \frac{\sqrt{2}\theta^2(1-\theta)}{\theta^2-4\theta+2}$. Computer calculation shows that the function $g'(x)$ has two real roots $x_1 = 0.313, x_2 = 5.83$. Therefore $g(x)$ increases for $0 \leq x \leq x_1$ and decreases for $x_1 \leq x \leq \frac{1}{3}$. Since $\theta' \leq \overline{\theta} \leq \frac{1}{3}$ then $g(\overline{\theta}) \geq \min(g(\theta'), g(\frac{1}{3})) = \min(0.08781\ldots, 0.199) = g(\theta')$.

Let $\{n_l\}$ be a sequence such that $\theta_{n_l} \to \overline{\theta}$. Then $g(\theta_{n_l}) \to g(\overline{\theta})$ and therefore

$$\overline{K} = \lim_{l\to\infty}\frac{\sum\limits_{k=1}^{K(n_l)} kx_k}{2^{n_l}n_l} \geq \lim_{l\to\infty} g(\theta_{n_l}) = g(\overline{\theta}) = 0.08781\ldots.$$

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References

[1] I. Akulich, A mind is good but five minds are better. *Kvant* **6** (1998), 11–16, (in Russian).

[2] A. Baababov, A "Pentium" is good but a mind is better. *Kvant* **4-5** (1999), 38–42, (in Russian).

[3] R. L. Graham, B. L. Rothchild, J. H. Spencer, *Ramsey theory*. John Wiley & sons, 1980.

[4] N. N. Konstantinov, N. B. Vasil'ev, A. K. Tolpygo, *Twelve tournaments*. Informational Center of International Mathematical Tournament of Towns, Moskow, 1991. (in Russian).

[5] G. A. Korn, T. M. Korn, *Mathematical Handbook*. McGraw-Hill, 1968.